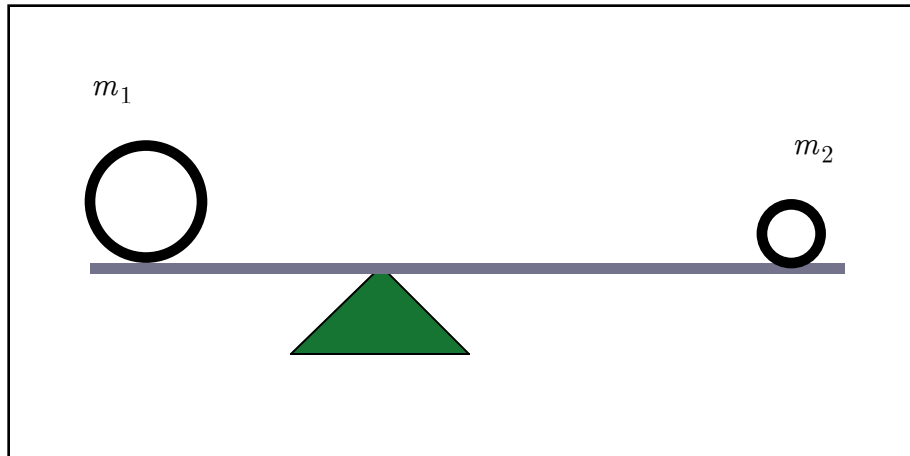


Moment and Centroids

Introduction to moment of inertia

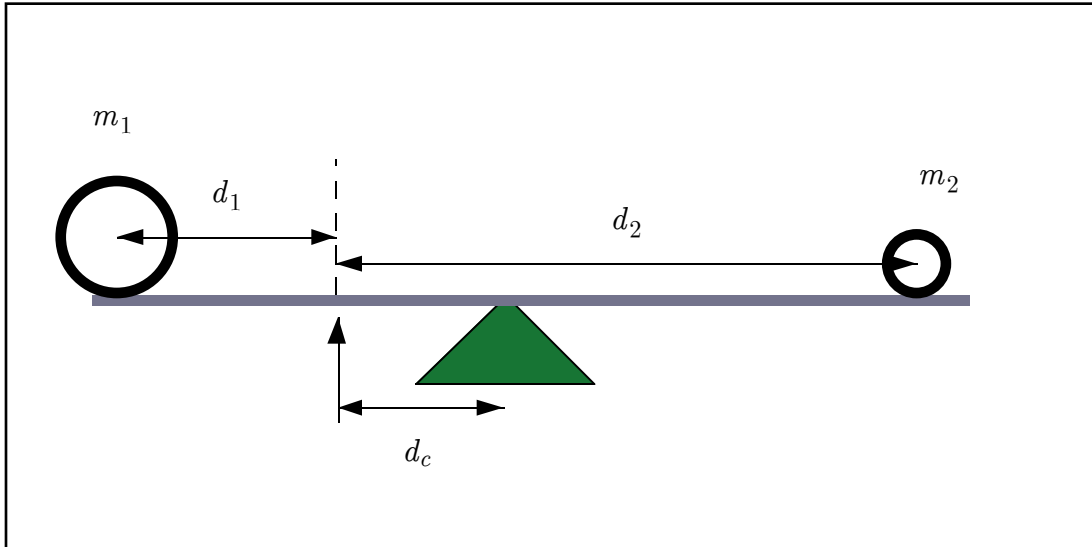
Consider two masses placed on a balance beam. How can we determine whether or not the masses will balance?



The answer is that each mass has associated with it a physical quantity called moment of inertia, or moment for short. The moment about a given point is defined as the product of the mass of the object times its distance from the point.

$$\text{moment} = \text{distance} * \text{mass}$$

Suppose we have two masses located on a line. How do we determine the balance point? The answer is that the balance point is that point with the property that if we place both masses at the point the total moment stays the same. This point is called the center of mass.



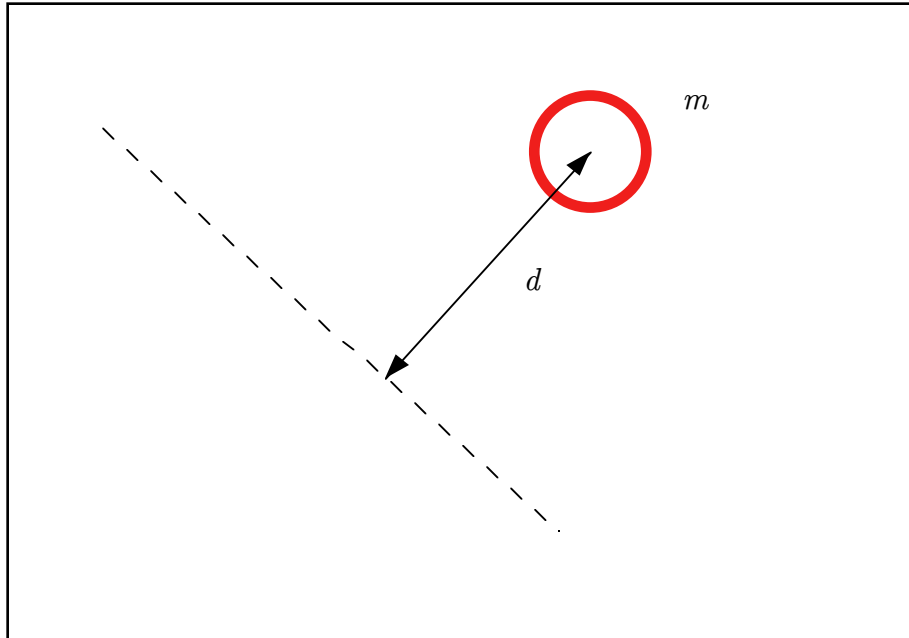
$$m_1 d_1 + m_2 d_2 = (m_1 + m_2) d_c$$

An important question about this definition has to do with its dependence on the reference point. Since the moment is defined in terms of the distance from the reference point, it seems quite reasonable to expect the location of the center of momentum to depend on the location of reference point. Fortunately, this is not the case.

If we move the reference point to a point x units away, every distance in the equation above gets modified by the same amount. The net result is that the center relative to the new reference point stays the same.

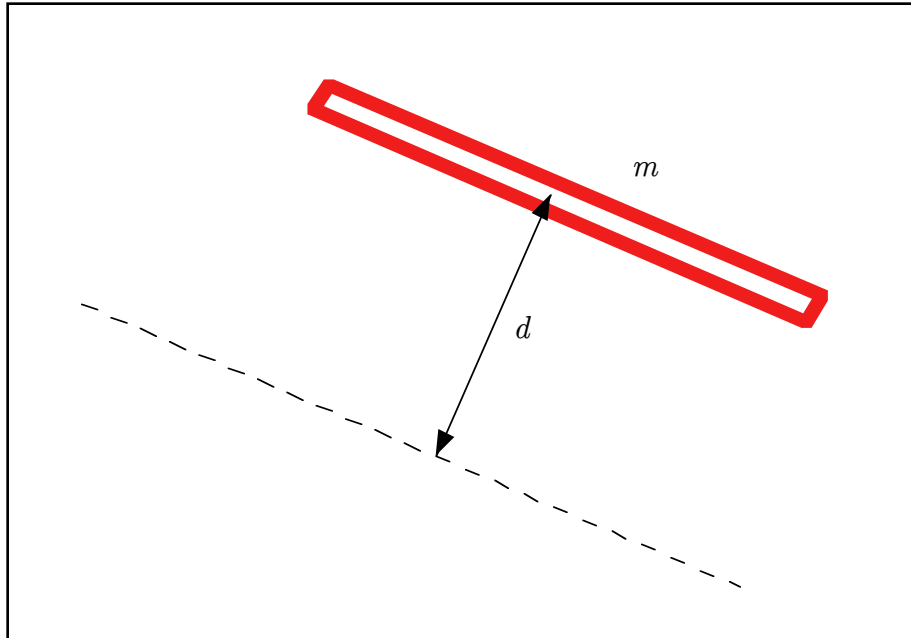
$$\begin{aligned} m_1 (d_1 - x) + m_2 (d_2 - x) &= m_1 d_1 + m_2 d_2 - (m_1 + m_2) x \\ &= (m_1 + m_2) d_c - (m_1 + m_2) x \\ &= (m_1 + m_2) (d_c - x) \end{aligned}$$

In two dimensions, we compute moments about lines. Again, the moment of an object about a given line is the product of its mass and its distance from the line.



The moment of an extended object about a line

The definition in the previous section applies to point masses. Our next concern is to extend this definition to extended, two dimensional objects. The key to this is to view the object as the sum of a large number of smaller masses. Since moment is additive, we can compute the moment of each component separately and then add up the moments to get a total. A further simplification comes from the observation that all of the masses that lie the same distance from the reference line can be combined into a rectangular mass.

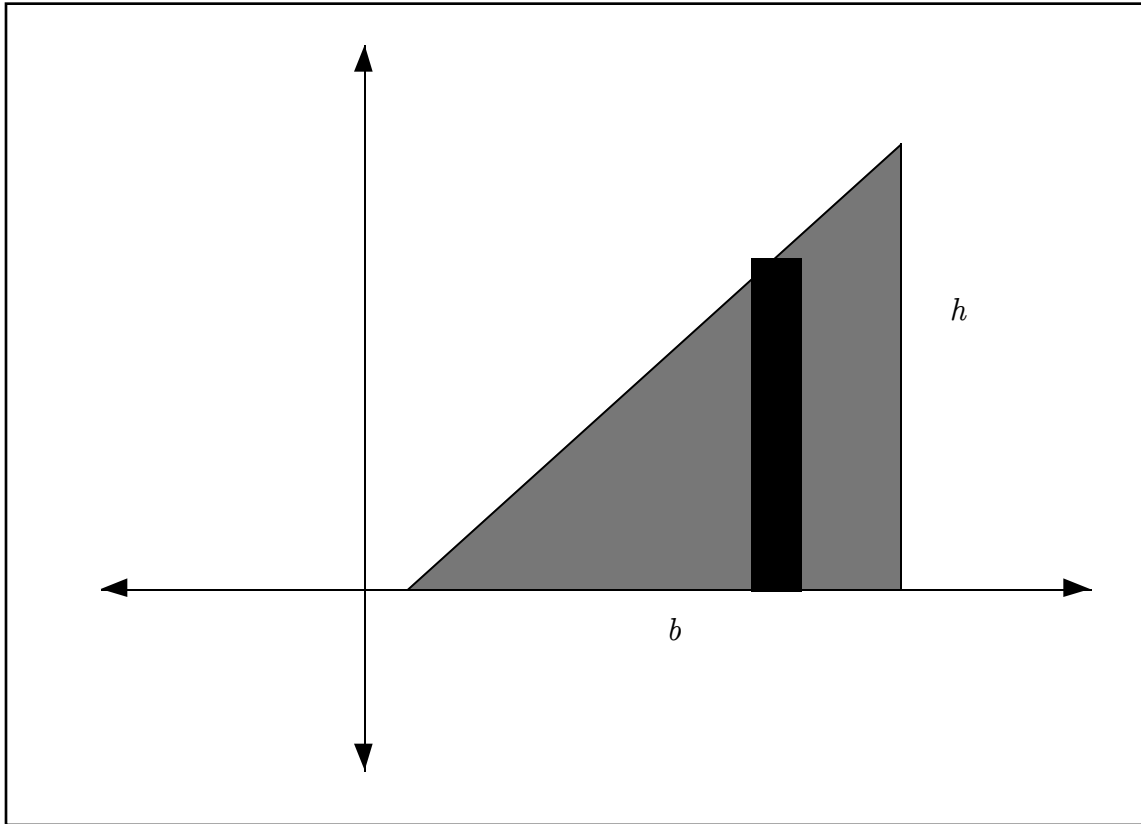


The moment of the rectangular strip pictured here is $m d$.

Given an extended object, we can compute its moment about a line by viewing the object as a collection of thin strips.

An example

Consider the right triangle depicted below. Suppose that the right triangle is constructed from a material with a uniform density of ρ units of mass per unit of area.



We are going to compute the moment of the triangle about the y axis. To simplify the calculation, we cut the triangle up into a set of thin vertical strips. We will compute the moment of each of the strips about the y axis and then integrate to add up all of the moments.

The moment of a strip centered at x is given by

$$\text{moment} = \text{mass} * \text{distance} = m(x) x$$

The mass of the strip, $m(x)$, is given by the density ρ times the area of the strip. The area is the product of the little bit of thickness dx and the height of the strip, $f(x)$.

$$m(x) = \rho A(x) = \rho f(x) dx$$

A simple argument shows that

$$f(x) = h/b x$$

Hence

$$\text{moment} = (\rho h/b x dx) x = \rho h/b x^2 dx$$

and

$$\text{total moment} = \rho h/b \int_0^b x^2 dx = \frac{1}{3} b^2 h \rho$$

To compute the distance from the reference line to the center of mass, we use the formula

$$\text{total moment} = (\text{total mass}) d_c$$

or

$$d_c = \frac{\text{total moment}}{\text{total mass}}$$

(The text calls the distance $d_c \bar{x}$.) The mass is easy to compute.

$$\text{total mass} = \rho \text{ area} = \rho 1/2 h b$$

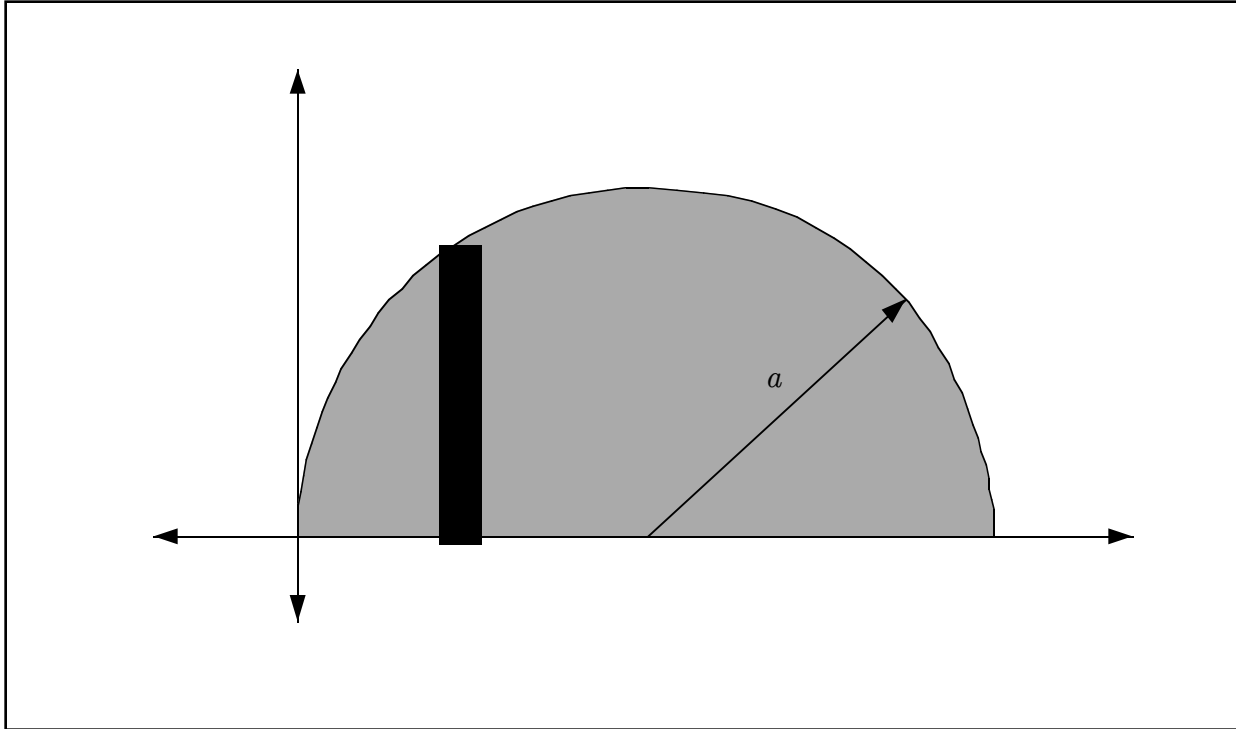
Thus

$$d_c = \frac{\frac{1}{3} b^2 h \rho}{\rho 1/2 h b} = \frac{2}{3} b$$

This says that if we lay a line parallel to the y -axis at $x = 2/3 b$ the triangle will balance exactly on the line. If you look back up at the picture of the triangle above, you will see that this answer appears to be perfectly reasonable.

Another example

This time we are going to compute the moment and centroid of the semicircle of radius a . The centroid of a plane figure is the point (\bar{x}, \bar{y}) at the intersection of the balancing lines parallel to the axes.



First we compute the moment relative to the y -axis and then use that to find the balancing line relative to the y -axis. Note that even without doing any calculation we can see by symmetry that the balancing line is located at $x = a$. The calculation below will confirm this.

A typical strip parallel to the reference line has a moment

$$\text{moment} = x m(x) = x \rho A(x)$$

We compute the area of the strip as the product of a height and a width.

$$A(x) = h(x) dx = \sqrt{a^2 - (a-x)^2} dx$$

Putting this all together gives us that

$$\text{total moment} = \rho \int_0^{2a} x \sqrt{a^2 - (a-x)^2} dx$$

You can do this integral via the substitution

$$u = a - x$$

which gives

$$\begin{aligned}
\text{total moment} &= \rho \int_a^{-a} (a - u) \sqrt{a^2 - u^2} (-1) du \\
&= -\rho \int_a^{-a} a \sqrt{a^2 - u^2} du + \rho \int_a^{-a} u \sqrt{a^2 - u^2} du \\
&= \rho \left(\frac{1}{4} a^3 \pi + \frac{1}{4} a^3 \pi \right) = \rho \frac{1}{2} a^3 \pi
\end{aligned}$$

The location of the balancing line (\bar{x}) is given by

$$d_c = \frac{\text{total moment}}{\text{total mass}} = \frac{\rho \frac{1}{2} a^3 \pi}{\rho \frac{1}{2} \pi a^2} = a$$

This is exactly the result we got by symmetry earlier.

Next, the moment of the figure relative to the x -axis. The strips parallel to the reference line are all horizontal strips. Their moment is given by

$$\text{moment} = y m(y) = y \rho A(y) = y \rho w(y) dy$$

To get the width $w(y)$ of the strips, we use the equation of the circle

$$(x - a)^2 + y^2 = a^2$$

to get the location of the endpoints of the strip at level y .

$$x = a \pm \sqrt{a^2 - y^2}$$

Thus

$$A(y) = \left((a + \sqrt{a^2 - y^2}) - (a - \sqrt{a^2 - y^2}) \right) dy = 2 \sqrt{a^2 - y^2} dy$$

and

$$\text{total moment} = 2 \rho \int_0^a y \sqrt{a^2 - y^2} dy$$

You can do this integral via the substitution

$$u = a^2 - y^2$$

$$total\ moment = 2 \rho \int_{a^2}^0 -\frac{1}{2} u^{\frac{1}{2}} du = 2 \rho \frac{1}{3} (a^2)^{\frac{3}{2}}$$

The location of the balancing line (\bar{y}) is

$$d_c = \frac{2 \rho \frac{1}{3} (a^2)^{\frac{3}{2}}}{\rho \frac{1}{2} \pi a^2} = \frac{4}{3 \pi} a = 0.42441318157838759 a$$

Next, an alternative way to compute the moment of this figure about the x-axis. The starting point for this method is to compute the moment of a vertical strip about the x-axis. We take advantage of the fact that one can write the moment of the strip as the product of its total mass times the distance to its center of mass.

$$moment = total\ mass * d_c$$

Because the vertical strip is such a simple object geometrically, we can say without calculating that its center of mass is located half way up the strip at $h(x)/2$.

$$moment = (\rho h(x) dx) (h(x)/2) = \frac{\rho}{2} (h(x))^2 dx$$

$$total\ moment = \int_0^{2a} \frac{\rho}{2} h^2(x) dx$$

In this case, the height of a vertical strip located at x is easy to compute from the equation of the circle.

$$h(x) = \sqrt{a^2 - (a - x)^2}$$

Quite conveniently, when we substitute $h(x)$ into the total moment integral the square root gets squared away and we end up with a simple integral.

$$total\ moment = \int_0^{2a} \frac{\rho}{2} (a^2 - (a - x)^2) dx = \frac{2}{3} a^3 \rho$$

This is exactly the same result we got from the horizontal strips calculation.